Partial entrainment in the Kuramoto model

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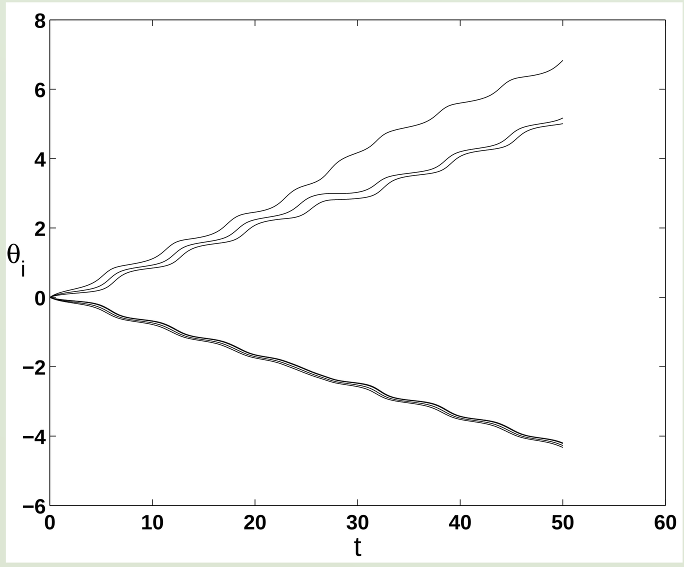
I. Introduction

- The Kuramoto model is a prototype model for investigating synchronization in systems of coupled oscillators. Examples of such systems are pacemaker cells in the heart, flashing fireflies, Josephson junctions,...
- The model is given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad \forall\, i \in \{1,\ldots,N\},$$

  where θ_i represents the phase of oscillator i.
- The individual frequencies ω_i are drawn from a distribution g.
- For zero coupling strength ($K = 0$) every oscillator moves at its own frequency ω_i.
- If the coupling strength is large enough oscillators can become entrained: their phase differences are bounded and thus the entrained oscillators move at the same long term average velocity (defined for oscillator i as $\lim_{t\to\infty}\frac{\Delta\theta_i(t)}{\Delta t} = \lim_{t\to\infty}\frac{\theta_i(t)-\theta_i(0)}{t}$).

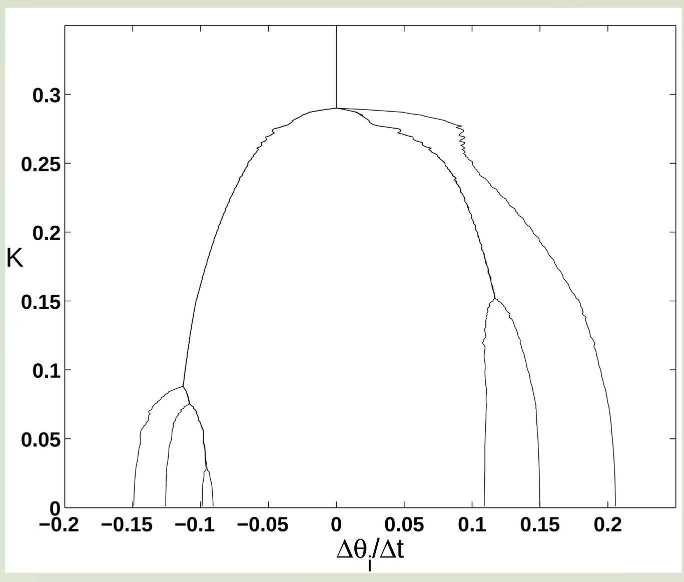


The time evolution of the phases θ_i is shown for a system with seven oscillators.

II. The finite N case

Simulation results:

- The entrainment behavior is independent of the initial condition.
- Different forms of partial entrainment occur, each characterized by an interval for the coupling strength K.
- With increasing K the entrainment behavior changes in a discrete way: in most of these changes two entrained subsets merge to form a new entrained subset.



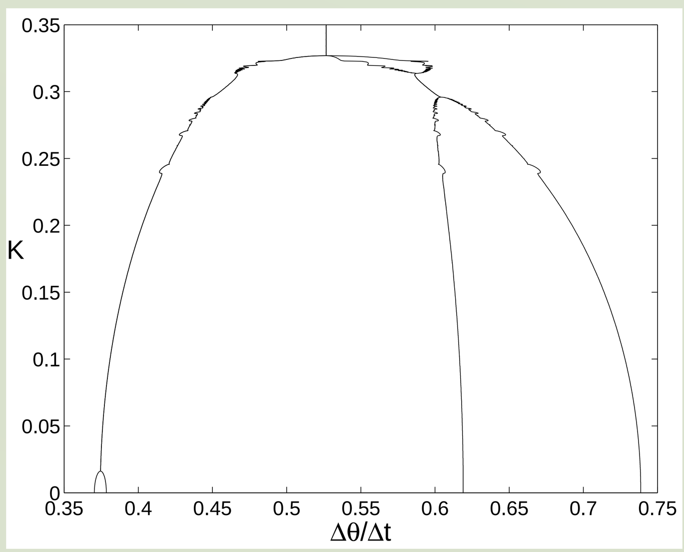
The long term average velocities (horizontal axis) are shown for varying coupling strength (vertical axis) for a system with seven oscillators. Different forms of partial entrainment are distinguishable: from no entrainment for small K, to full entrainment for large K.

Analytically:

- Sufficient conditions can be formulated that guarantee the existence of a solution with a particular partial entrainment behavior.
- For some systems this allows to analytically prove the occurrence of different forms of partial entrainment with varying coupling strength.

Simulation results:

- For some configurations the entrainment can disappear (temporarily) when K increases.



Long term average velocities (horizontal axis) are shown for varying coupling strength (vertical axis), for a system with four oscillators. When K is increased from 0.3 to 0.35 there is a transition in which entrainment of two oscillators disappears. It reappears at the transition to full entrainment.

III. The infinite N case

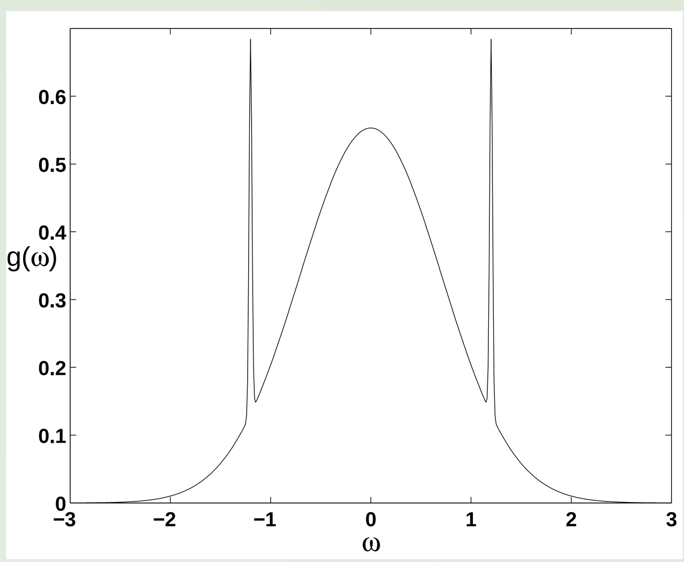
By investigating a perturbation of a known analytical solution it can be shown that:

1. Entrainment can *disappear* with increasing coupling strength.
2. Entrained subsets can induce entrainment of other oscillators.

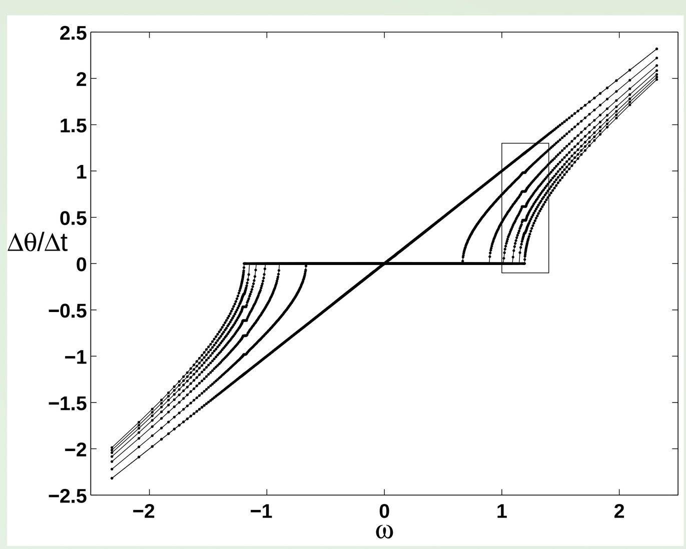
Example:

Define g by

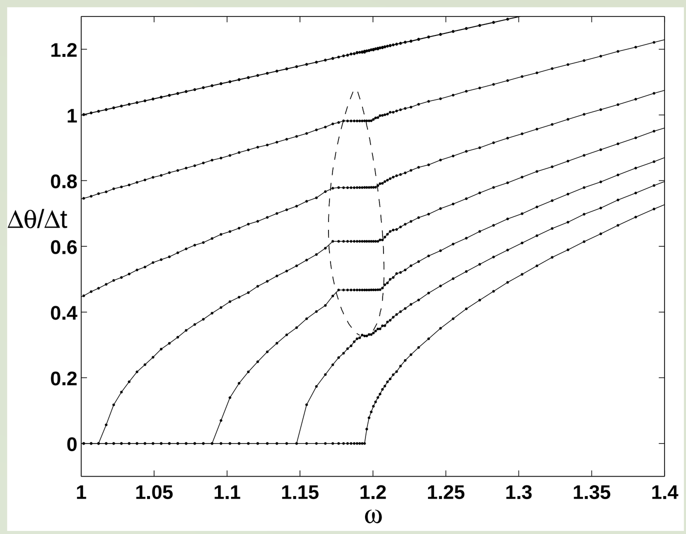
$$g(\omega) = \frac{1}{1.04\sqrt{\pi}}\left(e^{-\omega^2} + e^{-50^2(\omega-1.2)^2} + e^{-50^2(\omega+1.2)^2}\right).$$



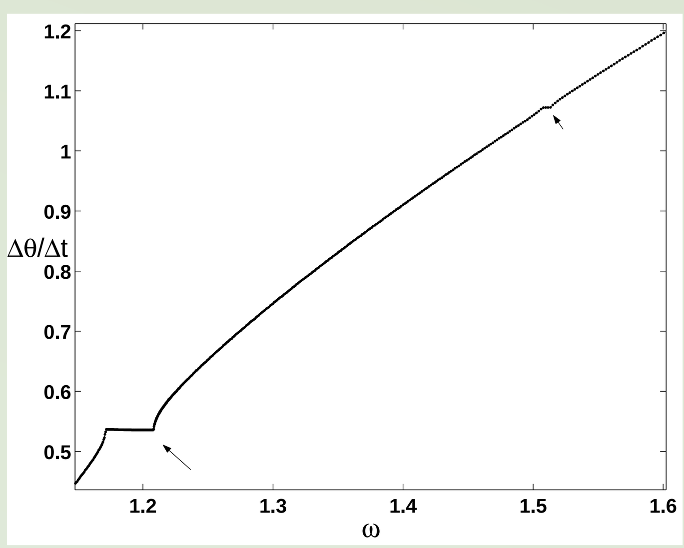
The distribution g.



The long term average velocities (vertical axis) are shown as a function of ω (horizontal axis) for different values of the coupling strength, as resulting from a simulation of a system of 1000 oscillators. The ω-values are chosen such that, when applying the cumulative density distribution associated with g, a linear division of the interval $(0,1)$ is obtained. The velocities decrease (in absolute value) with increasing K.



This picture is obtained by zooming in on the rectangular region indicated in the previous picture. The dashed line shows an analytical prediction of the boundaries of the entrained subsets. As is clearly visible, the length of the interval, corresponding to the entrained subset, first increases but then decreases again with increasing K, until the entrained subset has disappeared.



The long term average velocity is shown as a function of ω for $K = 1.5$, with $N = 10000$. The entrained subset with ω-values around 1.2 corresponds to one of the peaks of g. It induces entrainment of another subset at about $\omega \approx 1.5$, with an average velocity which is (exactly) twice as large.

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